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BRIEF COMMUNICATIONS

An Optimal Flow-Radius Equation for Microvessel Non-Newtonian Blood Flow

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The well known theoretical optimality relationship linking blood flow Q to microvessel radius r is given as $Q = kr^3$. As originally derived by Murray (1926) and subsequently applied to the microvasculature by Zamir (1976), Mayrovitz and Roy (1983), and others, a Newtonian rheology was intrinsically assumed. However, data for erythrocyte flow in 32- to 80-μm tubes (Goldsmith and Marlow, 1979) and *in vivo* measurements in microvessels (Rosenblum, 1972, Schmid-Schoenbein and Zweifach, 1975; Tangelder *et al.*, 1986) suggest a radial velocity profile significantly more blunt than the parabolic profile associated with pure Newtonian flow. It is thus of considerable interest to derive the applicable optimality relationship when non-Newtonian rheological behavior is considered.

A useful model for this purpose is the power-law fluid having the rheological equation $\tau = K_p \mid (\partial u/\partial r) \mid^{n-1} (\partial u/\partial r)^n$ in which u is the axial velocity and K_p is a constant corresponding to viscosity in the Newtonian case. Under steady laminar flow conditions in a vessel segment of radius R, length L, wall thickness h and axial pressure difference P, the axial velocity is given by (Hughes and Brighton, 1967)

$$u = (n/n + 1)(P/2k_pL)^{1/n} \{R^{(n+1)/n} - r^{(n+1)/n}\}. \tag{1}$$

The velocity profile associated with Eq. (1) contains the parabolic ($n = 1$) and plug-like flat profile ($n = 0$) as special cases. The corresponding volumetric flow rate is given by

$$Q = n\pi/(3n + 1) \{P/2k_p\}^{1/n} R^{(3n+1)/n}. \tag{2}$$

ANALYSIS

An optimality relationship is sought which minimizes the energy-equivalent cost of the sum of the hydraulic power H and a vascular volume partioned into blood volume B and wall volume W. The cost function C, is thus simply $H + k_bB + k_wW$ with the constants k_b and k_w in units of power/volume. In terms of the pertinent hemodynamic and vessel geometric parameters this cost function

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may be written as

$$C = QP + k_b(\pi r^2 L) + k_w(2\pi r L h) \tag{3}$$

By expressing P in terms of Q from Eq. (2) the result, $P = \{2K_p L(3n + 1)^n/(n\pi)^n\} \times \{Q^n/(r^{3n+1})\}$, may be used to write the cost per unit vessel length C_l as a function of Q as in

$$C_l = 2k_p\{(3n + 1)/n\pi\}^n Q^{n+1} r^{-(3n+1)} + k_b \pi r^2 + 2\pi k_w r h. \tag{4}$$

Prior to finding the minimum of C_l it is convenient to first approximate h in Eq. (4) by ξr. This approximation is based on the finding of an almost linear relationship between arteriolar radius and wall thickness where ξ is approximately 0.2 (Bohlen and Lobach, 1978). The resulting per unit vessel cost is then expressible as

$$C_l = Q^{n+1} 2k_p\{(3n + 1)/n\pi\}^n r^{-(3n+1)} + \pi(k_b + 2\xi k_w)r^2. \tag{5}$$

Equation (5) has the general form $C_l = Q^{n+1} A r^{-\alpha} + B r^{\beta}$ which when minimized with respect to r yields

$$Q^{n+1} = (B/A)(\beta/\alpha) r^{\alpha+\beta}. \tag{6}$$

For the present case $A = 2k_p\{(3n + 1)/n\pi\}^n$, $B = \pi(k_b + 2\xi k_w)$, $\alpha = 3n + 1$, and $\beta = 2$ and the optimality relationship between Q and r is obtained as

$$Q = \{(\pi/(3n + 1))(n(k_b + 2\xi k_w)/k_p)^{1/(n+1)}\} r^3. \tag{7}$$

The quantity within the braces includes several variables but for brevity will be denoted as $K(n)$. Using this notation we may write $Q = K(n)r^3$ with $K(n)$ given by

$$K(n) = (\pi/(3n + 1))(n(k_b + 2\xi k_w)/k_p)^{1/(n+1)}. \tag{8}$$

DISCUSSION

These results show that the third power dependence of flow on vessel radius ($Q = Kr^3$) which defines the optimal relationship for Newtonian flow also holds for the non-Newtonian power-law fluid. For the present more general case the value of K is seen to depend on n in a manner shown in Fig. 1 where the non-Newtonian to Newtonian ratio, $K(n)/K(1)$, is plotted. Using a viscosity of 0.028 P as was used by Murray (1926) and values for K_w and K_b of 59.9×10^3 and 4.34×10^3 erg cm^{-3} s^{-1} (Mayrovitz and Roy, 1983) the value obtained for K corresponding to the Newtonian case ($n = 1$) is $K(1) = 785$ s^{-1}.

The n dependent variation in K may explain a previously reported discrepency between the calculated value of K and that obtained from experimental data. From *in vivo* velocity–radius measurements in multiple arteriolar branching levels in the cremaster microvasculature (Mayrovitz and Roy, 1983) the equation $Q = 3359\ r^3$ was found to well describe the flow–radius relationship. However, the K value of 3359 s^{-1} based on the data was between 3.2 and 4.5 times larger than the theoretical optimal K value one would calculate based on Newtonian flow. The bounds on the theoretical value they calculated were 736–1056 s^{-1} which were close to the range of 652–1032 s^{-1} reported by Murray (1926) using different procedures. Based on the present results the discrepancy may have

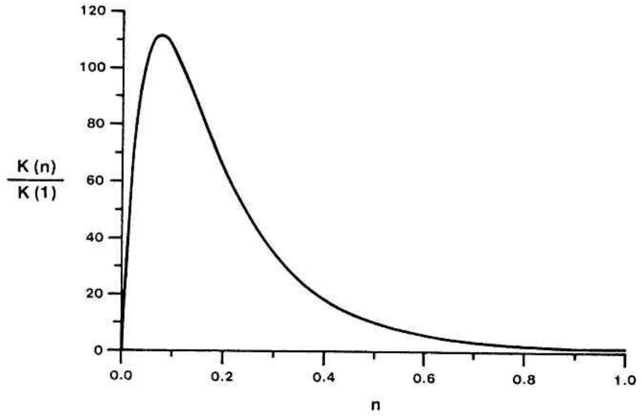

FIG. 1. Ratio of calculated non-Newtonian to Newtonian values for K in the optimality relationship $Q = Kr^3$. $K(n)$ is calculated from Eq. (8). For the Newtonian case, $n = 1$ and $K(1) = 785\ \mathrm{s}^{-1}$.

arisen because the Newtonian assumption in previous calculations was inappropriate. If non-Newtonian flow is assumed then one may calculate from Eq. (8) the value of n which would yield the experimentally determined value of K (3359). Carrying out this calculation with the previously mentioned parameter values, n is found to be approximately equal to 2/3. The solution for n is not very sensitive to the value of k_p, since over the k_p range from 0.020 to 0.040 n varies from 0.70 to 0.62, respectively. By using the mean velocity, $V = Q/(\pi R^2)$ to normalize the axial velocity the associated velocity profile (u/V) for $n = 2/3$ is obtained as shown in Fig. 2 with the parabolic profile $(n = 1)$ as a reference. If the actual average arteriolar *in vivo* profile is as shown for $n = 2/3$ rather than for $n = 1$ then the discrepancy between theory and experiment is reconciled. Verification of this is difficult since the original measurements (Mayrovitz and Roy, 1983) were made across an entire arteriolar microvasculature in vessels ranging from 83 to 7 μm. Thus the net K value they determined would intrinsically

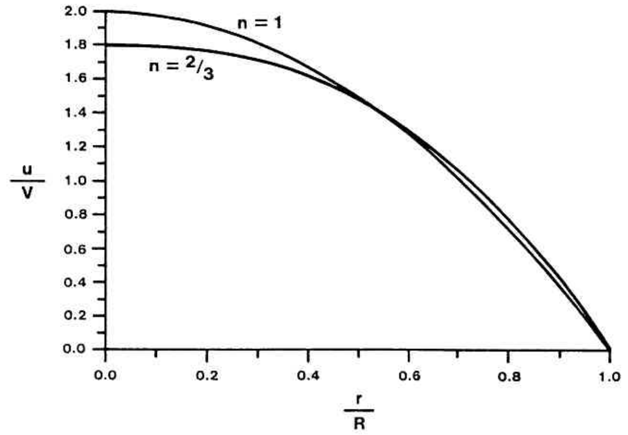

FIG. 2. Velocity profile corresponding to the experimentally determined value of $K(n = 2/3)$ compared with the Newtonian parabolic profile $(n = 1)$.

reflect the integrated contributions of velocity profiles in a wide range of vessel sizes, with variable hematocrit and mean blood velocities.

Data bearing on the nature of the profiles likely to be present are available. The detailed red blood cell velocity profiles systematically determined by Goldsmith and Marlow (1979) are particularly useful in estimating values of n which may characterize profiles in microvessels. They used ghost red cells as markers in 32- to 80-μm-radius tubes and presented the radial profiles in graphical form expressed as the ratio of u/u_m in which u_m was the maximum measured particle velocity in the tube central region. In all cases the profile was blunted as compared with a parabolic shape. Of the profiles presented by these authors, the one obtained under conditions most closely resembling average conditions in the arteriolar microvasculature had a red blood cell volume concentration of 0.43 in a tube with a radius of 42.6 μm. To estimate n from this plot, the power-law velocity profiles, $u/u_m = 1 - |(r/R)|^{(n+1)/n}$ was calculated and plotted to the same scale as enlargements of the figure of these authors. Using transparent overlays, the calculated profiles were compared with the published profiles and the best match determined by eye. Whereas, the match for $n = 2/3$ was judged to be good in the central region, the best overall matches (essentially overlapping within our resolution) were indistinguishible between $n = 0.5$ and 0.6. The difference in values of n (2/3 vs 0.5–0.6) can in part be explained by the value of viscosity used. The value of $n = 2/3$ predicted from the present analysis is based on a viscosity of 0.028 P as previously described. From Goldsmith and Marlow's data, a volume concentration of 0.43 (roughly equal to a systemic hematocrit value) corresponded to a viscosity of about 0.04 P. For this viscosity a value of $n = 0.62$ is calculated by the present analysis and is thus consistent with the published *in vitro* profile at a corresponding viscosity. Since the hematocrits found in the microvasculature tend to be lower than those measured in the systemic circulation, one would expect the *in vivo* value of n to be closer to the value of 2/3 herein obtained.

Recent measurements (Tangelder *et al.*, 1986) in arterioles (17–32 μm) appear to suggest an even smaller *in vivo* value of n. Using fluorescent platelets as markers, velocity profiles were obtained and curve fitted with nonlinear regression procedures and compared against parabolic profiles. Results from all seven vessels analyzed indicate profiles significantly blunter than prabolic. Results from fitting the power-law profile equation yielded a median value of the quantity $(n + 1)/n = 4.55$ indicating a value of $n = 0.28$. These authors note the presence of considerable scatter in the experimental data points through which the curves were fitted. They also note that the lack of data points in the vicinity of the vessel wall required a forcing of the velocity to zero to satisfy the no-slip condition. Subsequently they utilized a modified form of the power-law equation which excluded conditions within the vessel wall vicinity but resulted in a more uniform statistical residual pattern than with the strict power-law formulation. Using as a bluntness index the ratio of maximum to mean velocity (u_m/V), the median value they obtained was 1.5 thus indicating a blunter profile than for the corresponding value of 1.8 obtained for $n = 2/3$. However, these authors found a significant positive correlation between u_m/V and vessel diameter (D) given as $u_m/V = 0.009D + 1.24$. If the correlation holds beyond the diameter range for which they made their measurements, then the vessel diameter for which u_m/V

would be equal to 1.8 is calculated to be 62.2 μm. In summary, evidence from both *in vitro* and *in vivo* studies clearly have shown that for most, if not all, vessel sizes found within the microvasculature, the velocity profile is significantly flatter than the parabolic profile. Despite this, it has been shown that a law of optimality, previously based on the intrinsic assumption of a parabolic profile, is expressible in its original form $Q = Kr^3$ but that the value of the constant K is altered. A power-law exponent, $n = 2/3$, was determined to account for previously determined values of K and though this value of n is not inconsistent with available data it is not yet validated as an average characterization of the microvascular bed.

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